ADVANCED HIGHER MATHEMATICS

DIFFERENTIATION: THE PRODUCT RULE

1. Differentiate with respect to x, expressing each answer in its fully factorised form:

(a)	$y = x\sin x$	(b)	$y = x\cos x$	(c)	$y = x^2 \sin x$
(d)	$y = x^4 \cos x$	(e)	$y = x\sin 2x$	(f)	$y = x\cos 3x$
(g)	$y = x^3 \sin 4x$	(h)	$y = x\cos(x^2)$	(i)	$y = x^2 \sin(x^3)$
(j)	$y = x\cos 2x$	(k)	$y = \sin 2x \sin x$	(l)	$y = x^4 \sin 3x$
(m)	$y = x^2 \cos 2x$	(n)	$y = x\sin^2 x$	(o)	$y = x\cos^2 x$
(p)	$y = x\sin^3 x$	(q)	$y = x^2 \sin^2 x$	(r)	$y = 2x^2 \cos x$
(s)	$y = \sin 2x \cos 3x$	(t)	$y = x^2 \sin 2x$		

2. Given $f(x) = x\sin(\pi x)$, find the value of $f'\left(\frac{1}{2}\right)$.

3. Given $y = \sin x \cos x$, show that $\frac{dy}{dx} = \cos 2x$.

[*You may assume the trigonometric identity* $\cos 2x = \cos^2 x - \sin^2 x$.]

***4.** (a) Differentiate $\sin x \cos 2x$ with respect to x.

(b) Hence, or otherwise, find $\frac{dy}{dx}$ given $y = x^3 \sin x \cos 2x$.

5. Differentiate with respect to x, expressing each answer in its fully factorised form:

(a)	$y = x(x+1)^3$	(b)	$y = x(x-1)^4$	(c)	$y = x(x^2+1)^3$
(d)	$y = x^2(x+1)^4$	(e)	$y = x(x+2)^5$	(f)	$y = x(2x+1)^3$
(g)	$y = x(3x-1)^4$	(h)	$y = x^2(2x+3)^5$	(i)	$y = x^2(x^2+4)^3$
(j)	$y = x^3(x+2)^4$	(k)	$y = x^2(2x-1)^3$	(l)	$y = x^2(x^3+1)^4$
(m)	$y = x(2x+3)^3$	(n)	$y = (x+1)^2(x-1)^4$	(o)	$y = (x+1)^3(x+2)^4$
(p)	$y = (x+1)^4(x-1)^3$	(q)	$y = (x+1)^2(x+7)^5$	(r)	$y = (2x+1)^2(3x-1)^4$

6. Given $y = (x+3)^4(x-3)^5$, show that $\frac{dy}{dx} = 3(3x+1)(x+3)^3(x-3)^4$.

7. (a) Given $y = x^2(x-3)^4$, show that $\frac{dy}{dx} = 6x(x-1)(x-3)^3$.

(b) Find the coordinates and nature of each of the stationary points on the curve $y = x^2(x-3)^4$.

8. Differentiate with respect to x, expressing each answer as a single algebraic fraction in its simplest form:

(a)	$y = x\sqrt{x+1}$	(b)	$y = x\sqrt{x+4}$	(c)	$y = x\sqrt{2x+3}$
(d)	$y = x\sqrt{3x+1}$	(e)	$y = x\sqrt{x^2+1}$	(f)	$y = x^2\sqrt{x-1}$
(g)	$y = x\sqrt{x^2+4}$	(h)	$y = x^3\sqrt{x+1}$	(i)	$y = x^2\sqrt{x^2+1}$
(j)	$y = \sqrt{x}(x+1)^3$	(k)	$y = \sqrt{x}(x-1)^2$	(l)	$y = \sqrt{x}(x+2)^4$

ANSWERS

① (a) $x\cos x + \sin x$ (b) $\cos x - x\sin x$

(c) $x(x\cos x + 2\sin x)$ (d) $x^3(4\cos x - x\sin x)$

(e) $2x\cos 2x + \sin 2x$ (f) $\cos 3x - 3x\sin 3x$

(g) $2x(2x\cos 4x + \sin 4x)$ (h) $\cos(x^2) - 2x^2\sin(x^2)$

(i) $x\{3x^3\cos(x^3) + 2\sin(x^3)\}$ (j) $\cos 2x - 2x\sin 2x$

(k) $\sin 2x\cos x + 2\cos 2x\sin x$ (l) $x^3(3x\cos 3x + 4\sin 3x)$

(m) $2x(\cos 2x - x\sin 2x)$ (n) $\sin x(2x\cos x + \sin x)$

(o) $\cos x(\cos x - 2x\sin x)$ (p) $\sin^2 x(3x\cos x + \sin x)$

(q) $2x\sin x(x\cos x + \sin x)$ (r) $2x(2\cos x - x\sin x)$

(s) $2\cos 2x\cos 3x - 3\sin 2x\sin 3x$

(t) $2x(x\cos 2x + \sin 2x)$

② $f'\left(\frac{1}{2}\right) = 1$

④ (a) $\cos x\cos 2x - 2\sin x\sin 2x$

(b) $x^2(x\cos x\cos 2x - 2x\sin x\sin 2x + 3\sin x\cos 2x)$

⑤ (a) $(4x+1)(x+1)^2$ (b) $(5x-1)(x-1)^3$

(c) $(7x^2+1)(x^2+1)^2$ (d) $2x(3x+1)(x+1)^3$

(e) $2(3x+1)(x+2)^4$ (f) $(8x+1)(2x+1)^2$

(g) $(15x-1)(3x-1)^3$ (h) $2x(7x+3)(2x+3)^4$

(i) $8x(x^2+1)(x^2+4)^2$ (j) $x^2(7x+6)(x+2)^3$

(k) $2x(5x-1)(2x-1)^2$ (l) $2x(7x^3+1)(x^3+1)^3$

(m) $(8x+3)(2x+3)^2$ (n) $2(3x+1)(x+1)(x-1)^3$

(o) $(7x+10)(x+1)^2(x+2)^3$ (p) $(7x-1)(x+1)^3(x-1)^2$

(q) $(7x+19)(x+1)(x+7)^4$ (r) $4(9x+2)(2x+1)(3x-1)^3$

⑦ (b) min. T.P. $(0,0)$; max. T.P. $(1,16)$; min. T.P. $(3,0)$

⑧ (a) $\dfrac{3x+2}{2\sqrt{x+1}}$ (b) $\dfrac{3x+8}{2\sqrt{x+4}}$ (c) $\dfrac{3(x+1)}{\sqrt{2x+3}}$

(d) $\dfrac{9x+2}{2\sqrt{3x+1}}$ (e) $\dfrac{2x^2+1}{\sqrt{x^2+1}}$ (f) $\dfrac{x(5x-4)}{2\sqrt{x-1}}$

(g) $\dfrac{2(x^2+2)}{\sqrt{x^2+4}}$ (h) $\dfrac{x^2(7x+6)}{2\sqrt{x+1}}$ (i) $\dfrac{x(3x^2+2)}{\sqrt{x^2+1}}$

(j) $\dfrac{(7x+1)(x+1)^2}{2\sqrt{x}}$ (k) $\dfrac{(5x-1)(x-1)}{2\sqrt{x}}$ (l) $\dfrac{(9x+2)(x+2)^3}{2\sqrt{x}}$

DIFFERENTIATION: THE QUOTIENT RULE

1. Differentiate with respect to x, expressing each answer in its simplest form:

(a) $y=\dfrac{x}{x+4}$ (b) $y=\dfrac{3x}{x+2}$ (c) $y=\dfrac{x+1}{2x+1}$

(d) $y=\dfrac{x}{x^2+1}$ (e) $y=\dfrac{x^2}{x+3}$ (f) $y=\dfrac{x^2-1}{x^2+1}$

(g) $y=\dfrac{x^2}{4-x}$ (h) $y=\dfrac{2x-1}{3x+2}$ (i) $y=\dfrac{x^4}{x+1}$

(j) $y=\dfrac{x^2}{x^3+1}$ (k) $y=\dfrac{x^4}{x^2+1}$ (l) $y=\dfrac{2x^2}{x-2}$

(m) $y=\dfrac{\sin x}{x}$ (n) $y=\dfrac{x}{\cos x}$ (o) $y=\dfrac{x^2}{\sin x}$

(p) $y=\dfrac{x^3}{\sin x}$ (q) $y=\dfrac{\sin x}{x^2}$ (r) $y=\dfrac{x^4}{\cos x}$

(s) $y=\dfrac{\sin 2x}{x^2}$ (t) $y=\dfrac{\sin x}{\cos x}$ (u) $y=\dfrac{1+\sin x}{1+\cos x}$

(v) $y=\dfrac{\sin x}{\sin x+\cos x}$ (w) $y=\dfrac{x^3-1}{x^3+1}$ (x) $y=\dfrac{2x^2+3x-6}{x-2}$

2. Given $f(x)=\dfrac{x+1}{x^2+2}$, find the value of $f'(0)$.

3. Differentiate with respect to x, expressing each answer in its simplest form:

(a) $y=\dfrac{x}{(x+1)^2}$ (b) $y=\dfrac{x}{(x+2)^3}$ (c) $y=\dfrac{(x+1)^2}{(x+2)^3}$

(d) $y=\dfrac{(2x+1)^2}{(3x+1)^2}$

4. Differentiate with respect to x, expressing each answer in its simplest form:

(a) $y=\dfrac{x}{\sqrt{x+1}}$ (b) $y=\dfrac{x}{\sqrt{x-2}}$ (c) $y=\dfrac{x^2}{\sqrt{x+1}}$

(d) $y=\dfrac{x}{\sqrt{x^2+1}}$ (e) $y=\dfrac{x}{\sqrt{2x+1}}$ (f) $y=\dfrac{x^2}{\sqrt{x^3+1}}$

ANSWERS

1. (a) $\dfrac{4}{(x+4)^2}$ (b) $\dfrac{6}{(x+2)^2}$ (c) $-\dfrac{1}{(2x+1)^2}$

(d) $\dfrac{(1-x)(1+x)}{(x^2+1)^2}$ (e) $\dfrac{x(x+6)}{(x+3)^2}$ (f) $\dfrac{4x}{(x^2+1)^2}$

(g) $\dfrac{x(8-x)}{(4-x)^2}$ (h) $\dfrac{7}{(3x+2)^2}$ (i) $\dfrac{x^3(3x+4)}{(x+1)^2}$

(j) $\dfrac{x(2-x^3)}{(x^3+1)^2}$ (k) $\dfrac{2x^3(x^2+2)}{(x^2+1)^2}$ (l) $\dfrac{2x(x-4)}{(x-2)^2}$

(m) $\dfrac{x\cos x-\sin x}{x^2}$ (n) $\dfrac{\cos x+x\sin x}{\cos^2 x}$ (o) $\dfrac{x(2\sin x-x\cos x)}{\sin^2 x}$

(p) $\dfrac{x^2(3\sin x-x\cos x)}{\sin^2 x}$ (q) $\dfrac{x\cos x-2\sin x}{x^3}$ (r) $\dfrac{x^3(4\cos x+x\sin x)}{\cos^2 x}$

(s) $\dfrac{2(x\cos 2x-\sin 2x)}{x^3}$ (t) $\dfrac{1}{\cos^2 x}$ (u) $\dfrac{1+\cos x+\sin x}{(1+\cos x)^2}$

(v) $\dfrac{1}{(\sin x+\cos x)^2}$ (w) $\dfrac{6x^2}{(x^3+1)^2}$ (x) $\dfrac{2x(x-4)}{(x-2)^2}$

2. $f'(0)=\dfrac{1}{2}$

3. (a) $\dfrac{1-x}{(x+1)^3}$ (b) $\dfrac{2(1-x)}{(x+2)^4}$ (c) $\dfrac{(x+1)(1-x)}{(x+2)^4}$

(d) $-\dfrac{2(2x+1)}{(3x+1)^3}$

4. (a) $\dfrac{x+2}{2(x+1)^{3/2}}$ (b) $\dfrac{x-4}{2(x-2)^{3/2}}$ (c) $\dfrac{x(3x+4)}{2(x+1)^{3/2}}$

(d) $\dfrac{1}{(x^2+1)^{3/2}}$ (e) $\dfrac{x+1}{(2x+1)^{3/2}}$ (f) $\dfrac{x(x^3+4)}{2(x^3+1)^{3/2}}$

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

1. Use the chain rule to differentiate each function with respect to x:

(a)	$y = \sin 2x$	(b)	$y = \cos 4x$	(c)	$y = \tan 2x$
(d)	$y = \sec 3x$	(e)	$y = \operatorname{cosec} 5x$	(f)	$y = \cot 2x$
(g)	$y = \sin(x^2)$	(h)	$y = \cos(1 - 2x)$	(i)	$y = \tan 4x$
(j)	$y = \sec(2x + 3)$	(k)	$y = \operatorname{cosec} ax$	(l)	$y = \cot 3x$
(m)	$y = \sin^3 x$	(n)	$y = \cos^4 x$	(o)	$y = \tan^2 x$
(p)	$y = \sec^4 x$	(q)	$y = \operatorname{cosec}^2 x$	(r)	$y = \cot^3 x$
(s)	$y = \sin^2 3x$	(t)	$y = \cos^3 4x$	(u)	$y = \tan^3 2x$
(v)	$y = \sec^2 2x$	(w)	$y = \operatorname{cosec}^2 3x$	(x)	$y = \cot^2 5x$
(y)	$y = \tan^2 4x$	(z)	$y = \sec^6 3x$		

2. Use the product rule to differentiate each function with respect to x and simplify where possible:

(a)	$y = x \tan x$	(b)	$y = x^2 \sec x$	(c)	$y = x^2 \cot x$
(d)	$y = \sec x \tan x$	(e)	$y = x^2 \tan 2x$	(f)	$y = x^3 \operatorname{cosec} 2x$
(g)	$y = x^2 \sec 4x$	(h)	$y = x \tan^2 x$	(i)	$y = x^2 \sec^2 x$

3. Use the quotient rule to differentiate each function with respect to x and simplify where possible:

(a)	$y = \dfrac{x}{\tan x}$	(b)	$y = \dfrac{x}{\cot x}$	(c)	$y = \dfrac{\sec x}{x}$
(d)	$y = \dfrac{x^2}{\sec x}$	(e)	$y = \dfrac{x}{\operatorname{cosec} x}$	(f)	$y = \dfrac{\tan 2x}{x^2}$

4. Given $f(x) = \sin x \sec x$, show that $f'\left(\frac{\pi}{3}\right) = 4$.

[You can either use the product rule to differentiate $f(x)$ in the form above or you can simplify $f(x)$ before differentiating.]

ANSWERS

(1) (a) $2\cos 2x$ (b) $-4\sin 4x$

(c) $2\sec^2 2x$ (d) $3\sec 3x \tan 3x$

(e) $-5\operatorname{cosec} 5x \cot 5x$ (f) $-2\operatorname{cosec}^2 2x$

(g) $2x\cos(x^2)$ (h) $2\sin(1-2x)$

(i) $4\sec^2 4x$ (j) $2\sec(2x+3)\tan(2x+3)$

(k) $-a\operatorname{cosec} ax \cot ax$ (l) $-3\operatorname{cosec}^2 3x$

(m) $3\sin^2 x\cos x$ (n) $-5\cos^4 x\sin x$

(o) $2\sec^2 x\tan x$ (p) $4\sec^4 x\tan x$

(q) $-2\operatorname{cosec}^2 x\cot x$ (r) $-3\cot^2 x\operatorname{cosec}^2 x$

(s) $6\sin 3x\cos 3x$ (t) $-12\cos^2 4x\sin 4x$

(u) $6\tan^2 2x\sec^2 2x$ (v) $4\sec^2 2x\tan 2x$

(w) $-6\operatorname{cosec}^2 3x\cot 3x$ (x) $-10\operatorname{cosec}^2 5x\cot 5x$

(y) $8\sec^2 4x\tan 4x$ (z) $18\sec^6 3x\tan 3x$

(2) (a) $x\sec^2 x + \tan x$ (b) $x\sec x(x\tan x + 2)$

(c) $x(2\cot x - x\operatorname{cosec}^2 x)$ (d) $\sec x(\sec^2 x + \tan^2 x)$

(e) $2x(x\sec^2 2x + \tan 2x)$ (f) $x^2\operatorname{cosec} 2x(3 - 2x\cot 2x)$

(g) $2x\sec 4x(2x\tan 4x + 1)$ (h) $\tan x(2x\sec^2 x + \tan x)$

(i) $2x\sec^2 x(x\tan x + 1)$

(3) (a) $\dfrac{\tan x - x\sec^2 x}{\tan^2 x}$ (b) $\dfrac{\cot x + x\operatorname{cosec}^2 x}{\cot^2 x}$

(c) $\dfrac{\sec x(x\tan x - 1)}{x^2}$ (d) $\dfrac{x(2 - x\tan x)}{\sec x}$

(e) $\dfrac{1 + x\cot x}{\operatorname{cosec} x}$ (f) $\dfrac{2(x\sec^2 2x - \tan 2x)}{x^3}$

DIFFERENTIATION OF EXPONENTIAL FUNCTIONS

1. Use the chain rule to differentiate each function with respect to x:

(a) $y = e^{4x}$ (b) $y = e^{x^2}$ (c) $y = e^{\sin x}$

(d) $y = 2e^{3x}$ (e) $y = e^{\ 2x}$ (f) $y = e^{x^3}$

(g) $y = 8e^{\frac{1}{2}x}$ (h) $y = e^{\cos x}$ (i) $y = 4e^{-3x}$

(j) $y = 2e^{\ x}$ (k) $y = e^{-3x^2}$ (l) $y = e^{\sin 2x}$

(m) $y = e^{2x^2+1}$ (n) $y = 4e^{2x} + 3e^{-4x}$ (o) $y = 3e^{x^2} - 5e^{-2x}$

(p) $y = e^{\sqrt{x}}$

2. Use the product rule to differentiate each function with respect to x and simplify:

(a) $y = xe^{x}$ (b) $y = e^{x}\sin x$ (c) $y = x^3e^{x}$

(d) $y = xe^{2x}$ (e) $y = e^{-2x}\sin x$ (f) $y = x^2e^{4x}$

(g) $y = 2x^3e^{2x}$ (h) $y = e^{x}\cos 2x$ (i) $y = xe^{x^2}$

(j) $y = x^2e^{-2x}$ (k) $y = e^{2x}\sin 4x$ (l) $y = x^3e^{3x}$

(m) $y = (2x-1)e^{2x}$

3. Use the quotient rule to differentiate each function with respect to x and simplify:

(a) $y = \dfrac{e^x}{x}$ (b) $y = \dfrac{e^{2x}}{x}$ (c) $y = \dfrac{x}{e^x}$

(d) $y = \dfrac{e^x}{x+1}$ (e) $y = \dfrac{e^{2x}}{x^2}$ (f) $y = \dfrac{\sin x}{e^x}$

(g) $y = \dfrac{e^x}{x^3}$ (h) $y = \dfrac{e^x}{\sin x}$ (i) $y = \dfrac{x^2}{e^x}$

(j) $y = \dfrac{e^{2x}}{\cos x}$ (k) $y = \dfrac{x^2}{e^{3x}}$ (l) $y = \dfrac{e^{2x}-1}{e^{2x}+1}$

(m) $y = \dfrac{x(x+2)}{e^x}$

ANSWERS

1. (a) $4e^{4x}$ (b) $2xe^{x^2}$ (c) $\cos x\, e^{\sin x}$
   (d) $6e^{3x}$ (e) $-2e^{-2x}$ (f) $3x^2e^{x^3}$
   (g) $4e^{\frac{1}{2}x}$ (h) $-\sin x\, e^{\cos x}$ (i) $-12e^{-3x}$
   (j) $-2e^{-x}$ (k) $-6xe^{-3x^2}$ (l) $2\cos 2x\, e^{\sin 2x}$
   (m) $4xe^{2x^2+1}$ (n) $8e^{2x}-12e^{-4x}$ (o) $6xe^{x^2}+10e^{-2x}$
   (p) $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

2. (a) $(x+1)e^{x}$ (b) $e^{x}(\cos x+\sin x)$
   (c) $x^2(x+3)e^{x}$ (d) $(2x+1)e^{2x}$
   (e) $e^{-2x}(\cos x-2\sin x)$ (f) $2x(2x+1)e^{4x}$
   (g) $2x^2(2x+3)e^{2x}$ (h) $e^{x}(\cos 2x-2\sin 2x)$
   (i) $(2x^2+1)e^{x^2}$ (j) $2x(1-x)e^{-2x}$
   (k) $2e^{2x}(2\cos 4x+\sin 4x)$ (l) $3x^2(x+1)e^{3x}$
   (m) $4xe^{2x}$

3. (a) $\dfrac{(x-1)e^{x}}{x^2}$ (b) $\dfrac{(2x-1)e^{2x}}{x^2}$
   (c) $\dfrac{1-x}{e^{x}}$ (d) $\dfrac{xe^{x}}{(x+1)^2}$
   (e) $\dfrac{2(x-1)e^{2x}}{x^3}$ (f) $\dfrac{\cos x-\sin x}{e^{x}}$
   (g) $\dfrac{(x-3)e^{x}}{x^4}$ (h) $\dfrac{e^{x}(\sin x-\cos x)}{\sin^2 x}$
   (i) $\dfrac{x(2-x)}{e^{x}}$ (j) $\dfrac{e^{2x}(2\cos x+\sin x)}{\cos^2 x}$
   (k) $\dfrac{x(2-3x)}{e^{3x}}$ (l) $\dfrac{4e^{2x}}{(e^{2x}+1)^2}$
   (m) $\dfrac{2-x^2}{e^{x}}$

DIFFERENTIATION OF LOGARITHMIC FUNCTIONS

1. Use the chain rule to differentiate each function with respect to x and simplify where possible:

(a) $y = \ln(2x+1)$	(b) $y = \ln(4x+1)$	(c) $y = \ln(x^2+1)$
(d) $y = \ln(x-1)$	(e) $y = \ln(x^4+1)$	(f) $y = \ln(\cos x)$
(g) $y = \ln(2x^2+1)$	(h) $y = \ln(1-2x)$	(i) $y = \ln(\sin x)$
(j) $y = \ln(3x+2)$	(k) $y = \ln(2x^3+1)$	(l) $y = \ln(\cos 2x)$
(m) $y = \ln(e^x+1)$	(n) $y = \ln(\ln x)$	(o) $y = \ln(\sec x)$
(p) $y = \ln(\sec x + \tan x)$	(q) $y = (\ln x)^2$	(r) $y = \ln(\sin 4x)$

2. Use the product rule or quotient rule to differentiate each function with respect to x and simplify:

(a) $y = x\ln x$	(b) $y = e^x \ln x$	(c) $y = x^2 \ln x$
(d) $y = \dfrac{\ln x}{x}$	(e) $y = \dfrac{x}{\ln x}$	(f) $y = \dfrac{\ln x}{e^x}$
(g) $y = \dfrac{x^2}{\ln x}$	(h) $y = \dfrac{\ln x}{x^3}$	(i) $y = x^4 \ln x$
(j) $y = e^{2x}\ln x$	(k) $y = \dfrac{e^x}{\ln x}$	

3. Using the laws of logarithms first, differentiate each function with respect to x and simplify where possible:

(a) $y = \ln(x^2 e^x)$	(b) $y = \ln\sqrt{2x+1}$	(c) $y = \ln\left(\dfrac{x}{x+1}\right)$
(d) $y = \ln(x^3 e^{2x})$	(e) $y = \ln\sqrt{x^2+1}$	(f) $y = \ln\left(\dfrac{x^2}{2x+1}\right)$
(g) $y = \ln(x\cos x)$	(h) $y = \ln\sqrt{x^4+1}$	(i) $y = \ln\left(\dfrac{e^{2x}}{x^4}\right)$

4. (a) Given $y = \ln\left(\dfrac{x-1}{x+1}\right)$, show that $\dfrac{dy}{dx} = \dfrac{2}{x^2-1}$.

(b) Given $y = \ln\left(\dfrac{2x-1}{2x+1}\right)$, show that $\dfrac{dy}{dx} = \dfrac{4}{4x^2-1}$.

***5.** The function $y = 3^x$ can be differentiated with respect to x using the following method:

$$\begin{aligned} & y = 3^x \\ \Rightarrow\ & \ln y = \ln(3^x) \\ \Rightarrow\ & \ln y = x\ln 3 \\ \Rightarrow\ & x\ln 3 = \ln y \\ \Rightarrow\ & x = \frac{1}{\ln 3}\cdot \ln y \end{aligned}$$

Differentiate both sides with respect to y:

$$\frac{dx}{dy} = \frac{1}{\ln 3}\cdot\frac{1}{y} = \frac{1}{y\ln 3}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = y\ln 3 = 3^x \times \ln 3$$

Hence $\dfrac{dy}{dx} = 3^x \times \ln 3$.

Use a similar method to differentiate each of these functions with respect to x:

(a) $y = 2^x$ (b) $y = 10^x$ (c) $y = 4^{2x}$

***6.** The function $y = \log_{10} x$ can be differentiated with respect to x using the following method:

$$\begin{aligned} & y = \log_{10} x \\ \Rightarrow\ & 10^y = x \\ \Rightarrow\ & \ln(10^y) = \ln x \\ \Rightarrow\ & y\ln 10 = \ln x \\ \Rightarrow\ & y = \frac{1}{\ln 10}\cdot \ln x \end{aligned}$$

Differentiate with respect to x:

$$\frac{dy}{dx} = \frac{1}{\ln 10}\cdot\frac{1}{x} = \frac{1}{x\ln 3}$$

✱ P.T.O.

Hence $\frac{dx}{dy} = \frac{1}{x \ln 3}$.

Use a similar method to differentiate each of these functions with respect to x:

(a) $y = \log_2 x$ (b) $y = \log_3 x$ (c) $y = \log_5 x$

HIGHER DERIVATIVES

1. Find the second derivative $\frac{d^2y}{dx^2}$ in each case:

(a) $y = x^3$ (b) $y = 2x^6$ (c) $y = 4x^2$

(d) $y = 3x^5$ (e) $y = \sin x$ (f) $y = e^{3x}$

(g) $y = 3x^4 + 2x^3 + 4x^2 + 5x + 1$ (h) $y = x^3 - 2x^2 + 6x + 4$

(i) $y = \sin 2x$ (j) $y = (x+1)^5$ (k) $y = (2x+1)^6$

(l) $y = (3x-1)^4$ (m) $y = 3e^{2x} + 4e^{-2x}$ (n) $y = 4\cos 2x$

(o) $y = 3x^3 - 6x + 4$ (p) $y = (4x+1)^{\frac{3}{2}}$ (q) $y = \sqrt{2x-1}$

(r) $y = \ln x$ (s) $y = \ln(\cos x)$

2. Given $y = (x^2+1)^4$, show that $\frac{d^2y}{dx^2} = 8(7x^2+1)(x^2+1)^2$.

3. Given $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} = 2e^x \cos x$.

4. Given $y = \sin(x^2)$, show that $\frac{d^2y}{dx^2} = 2\left\{\cos(x^2) - 2x^2\sin(x^2)\right\}$.

5. Given $y = x^2 \ln x$, show that $\frac{d^2y}{dx^2} = 3 + 2\ln x$.

6. Given $y = e^{x^2}$, show that $\frac{d^2y}{dx^2} = 2(2x^2+1)e^{x^2}$.

7. Given $y = x\sin x$, show that $\frac{d^2y}{dx^2} = 2\cos x - x\sin x$.

8. Given $y = \frac{x}{2x+1}$, show that $\frac{d^2y}{dx^2} = -\frac{4}{(2x+1)^3}$.

9. Given $y = e^{ax}$, where a is a constant, find expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

Conjecture an expression for the n^{th} derivative $\frac{d^ny}{dx^n}$.

10. Given $y = xe^x$, show that $\frac{dy}{dx} = (x+1)e^x$ and obtain similar expressions for $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

Conjecture an expression for the n^{th} derivative $\frac{d^ny}{dx^n}$.

11. Given $y = x^2e^x$, show that $\frac{d^2y}{dx^2} = (x^2+4x+2)e^x$.

ANSWERS

1. (a) $12x^2$ (b) $60x^4$ (c) 8
   (d) $60x^3$ (e) $-\sin x$ (f) $9e^{3x}$
   (g) $36x^2+12x+8$ (h) $6x-4$ (i) $-4\sin 2x$
   (j) $20(x+1)^3$ (k) $120(2x+1)^4$ (l) $108(3x-1)^2$
   (m) $12e^{2x}+16e^{-2x}$ (n) $-16\cos 2x$ (o) $18x$
   (p) $\dfrac{12}{\sqrt{4x+1}}$ (q) $-\dfrac{1}{(2x-1)^{3/2}}$ (r) $-\dfrac{1}{x^2}$
   (s) $-\sec^2 x$

9. $\dfrac{dy}{dx} = ae^{ax}$, $\dfrac{d^2y}{dx^2} = a^2e^{ax}$, $\dfrac{d^3y}{dx^3} = a^3e^{ax}$;
   $\dfrac{d^ny}{dx^n} = a^ne^{ax}$

10. $\dfrac{d^2y}{dx^2} = (x+2)e^x$, $\dfrac{d^3y}{dx^3} = (x+3)e^x$;
    $\dfrac{d^ny}{dx^n} = (x+n)e^x$

ANSWERS

1. (a) $\dfrac{2}{2x+1}$ (b) $\dfrac{4}{4x+1}$ (c) $\dfrac{2x}{x^2+1}$

(d) $\dfrac{1}{x-1}$ (e) $\dfrac{4x^3}{x^4+1}$ (f) $-\tan x$

(g) $\dfrac{4x}{2x^2+1}$ (h) $-\dfrac{2}{1-2x}$ (i) $\cot x$

(j) $\dfrac{3}{3x+2}$ (k) $\dfrac{6x^2}{2x^3+1}$ (l) $-2\tan 2x$

(m) $\dfrac{e^x}{e^x+1}$ (n) $\dfrac{1}{x\ln x}$ (o) $\tan x$

(p) $\sec x$ (q) $\dfrac{2\ln x}{x}$ (r) $4\cot 4x$

2. (a) $1+\ln x$ (b) $\dfrac{e^x(1+x\ln x)}{x}$

(c) $x(1+2\ln x)$ (d) $\dfrac{1-\ln x}{x^2}$

(e) $\dfrac{\ln x-1}{(\ln x)^2}$ (f) $\dfrac{1-x\ln x}{xe^x}$

(g) $\dfrac{x(2\ln x-1)}{(\ln x)^2}$ (h) $\dfrac{1-3\ln x}{x^4}$

(i) $x^3(1+4\ln x)$ (j) $\dfrac{e^{2x}(1+2x\ln x)}{x}$

(k) $\dfrac{e^x(x\ln x-1)}{x(\ln x)^2}$

3. (a) $\dfrac{2}{x}+1$ (b) $\dfrac{1}{2x+1}$

(c) $\dfrac{1}{x(x+1)}$ (d) $\dfrac{3}{x}+2$

(e) $\dfrac{x}{x^2+1}$ (f) $\dfrac{2(x+1)}{x(2x+1)}$

(g) $\dfrac{1}{x}-\tan x$ (h) $\dfrac{2x^3}{x^4+1}$

(i) $2-\dfrac{4}{x}$

5. (a) $2^x\times\ln 2$ (b) $10^x\times\ln 10$ (c) $2\times 4^{2x}\times\ln 4$

6. (a) $\dfrac{1}{x\ln 2}$ (b) $\dfrac{1}{x\ln 3}$ (c) $\dfrac{1}{x\ln 5}$

ADVANCED HIGHER MATHEMATICS

HOMEWORK ON DIFFERENTIATION

1. Find the derivative of the function $f(x) = 2x^2 - 3x + 1$ **from first principles**.

2. Differentiate with respect to x and simplify:

(a) $y = x^3 \sin x$ (b) $y = x^4 \cos 2x$ (c) $y = x \sin^2 x$

(d) $y = x^2(x^2+1)^4$ (e) $y = x\sqrt{x-1}$ (f) $y = \dfrac{x}{1-2x}$

(g) $y = \dfrac{x^2}{x^2+1}$ (h) $y = \dfrac{\sin x}{x^2}$ (i) $y = \dfrac{\cos x}{\sin x + \cos x}$

(j) $y = \dfrac{x}{\sqrt{x^2+1}}$

3. Differentiate with respect to x and simplify:

(a) $y = \sec x \tan x$ (b) $y = \dfrac{x^2}{\operatorname{cosec} x}$ (c) $y = e^{\cot 4x}$

(d) $y = \ln(\sec 2x)$

4. Differentiate with respect to x and simplify:

(a) $y = xe^{-2x}$ (b) $y = \dfrac{e^{2x}}{x^2}$ (c) $y = \dfrac{e^{3x}-1}{e^{3x}+1}$

(d) $y = \dfrac{\ln x}{x^3}$ (e) $y = \ln\left(\dfrac{x^2}{x^2+1}\right)$ (f) $y = \ln\sqrt{x^2+4}$

5. Find the second derivative $\dfrac{d^2y}{dx^2}$ in its simplest form for each function below:

(a) $y = x^3 \ln x$ (b) $y = e^{2x} \sin 2x$