ADVANCED HIGHER MATHEMATICS

COMPLEX NUMBERS 10

MISCELLANEOUS PROBLEMS

1. Express each of the following complex numbers in the form $x+yi$, where x and y are real numbers.

(a) $(1-2i)(5+i)$	(b) $(2+i)^2$	(c) $\frac{1+i}{1-i}$
(d) $\frac{2+3i}{4-i}$	(e) $\frac{1}{i-2}$	(f) $\frac{3+2i}{3-2i}$
(g) $\frac{1}{(1+i)(2-i)}$	(h) $\frac{(2+i)(2+3i)}{3+4i}$	

2. Given $z=3+4i$, express $z+\frac{25}{z}$ in the form $a+bi$, where a and b are real numbers.

3. If $(5+i)z=3-2i$, find z in the form $p+qi$, where p and q are real numbers.

4. Let $z_1=\frac{2-i}{2+i}$ and $z_2=\frac{2i-1}{1-i}$.
   (a) Express z_1 and z_2 in the form $x+yi$, where x and y are real numbers.
   (b) Sketch an Argand diagram showing the complex numbers $5z_1+2z_2$ and $5z_1-2z_2$.

5. Given $(1+3i)z=5(1+i)$, express z and z^2 in the form $x+yi$, where x and y are real numbers.

6. Let $z=\frac{5+12i}{3+4i}$.
   (a) Express z in the form $x+yi$, where x and y are real numbers.
   (b) Hence find the modulus and argument (correct to the nearest $0.1°$) of z.

7. Find $(2-i)^3$, expressing your answer in the form $a+bi$, where a and b are real numbers.

8. The complex number z is such that $\frac{z}{z+2}=2-i$.
   (a) Find z in the form $x+yi$, where x and y are real numbers.
   (b) Find the modulus and argument (correct to the nearest $0.1°$) of z.

9. Find the two roots of the quadratic equation $z^2-4z+20=0$, expressing each root in the form $a+bi$, where a and b are real numbers.

   Find the modulus and argument (correct to the nearest $0.1°$) of each root.

10. The complex numbers z_1 and z_2 defined by $z_1=\frac{a}{1+i}$ and $z_2=\frac{b}{1+2i}$, where a and b are real numbers, are such that $z_1+z_2=1$.

    By expressing z_1 and z_2 in the form $x+yi$, where x and y are real numbers, or otherwise, find the values of a and b.

11. Let $z_1=1-i$, $z_2=4+3i$ and $z_3=z_1z_2$.
    (a) Express z_3 in the form $x+yi$, where x and y are real numbers.
    (b) Find the modulus and argument (correct to the nearest $0.1°$ where necessary) of: (i) z_1 (ii) z_2 (iii) z_3.
    (c) The complex numbers z_1, z_2 and z_3 are represented on an Argand diagram by the points A, B and C respectively.
    Show the complex numbers z_1, z_2 and z_3 on a single Argand diagram and find the area of triangle ABC.

12. Let $z_1=(1-i)(1+2i)$, $z_2=\frac{2+6i}{3-i}$ and $z_3=\frac{-4i}{1-i}$.
    (a) Express each of z_1, z_2 and z_3 in the form $a+bi$, where a and b are real numbers.
    (b) The complex numbers α and β are defined by $\alpha=z_2-z_1$ and $\beta=z_1-z_3$.
    Show that the modulus of α is equal to the modulus of β.

13. Express $\frac{1}{(3+2i)^2}$ in the form $x+yi$, where x and y are real numbers.

14. Let $z=2+\cos\theta+i\sin\theta$, where θ is a real number.

    Prove that the modulus of z is given by $(5+4\cos\theta)^{\frac{1}{2}}$.

15. Let $z_1=1-i$ and $z_2=7+i$.

    The complex number w is defined by $w=\frac{z_1-z_2}{z_1z_2}$.
    (a) Express w in the form $a+bi$, where a and b are real numbers.
    (b) Find the modulus of w in the form $k\sqrt{10}$, where k is a real number.

16. Express the complex numbers $1+i$ and $1-i$ in polar form.
    Hence simplify $(1+i)^{20}+(1-i)^{20}$.

17. Express the complex number $\sqrt{3}+i$ in polar form.
Hence find the smallest positive integer n for which $(\sqrt{3}+i)^n$ is:
(a) real (b) pure imaginary.

18. Find the roots of the equation $z^2+4z+8=0$, expressing each root in the form $x+yi$, where x and y are real numbers.

If the roots are denoted by α and β, simplify as far as possible the expression
$$\frac{\alpha+\beta+4i}{\alpha\beta+8i}.$$

19. The complex number z and its conjugate $\bar{z}$ satisfy the equation
$$z\bar{z}+2iz=12+6i.$$
Find the two possible complex numbers z by writing $z=x+yi$, where x and y are real numbers, and equating real and imaginary parts.

20. Find a real root of the cubic equation $z^3-4z^2+9z-10=0$ and hence solve this equation completely.

21. (a) Express $8i$ in the form $r(\cos\theta+i\sin\theta)$, where $r>0$ and $-180°<\theta\le 180°$.
(b) Find the roots of the equation $z^3=8i$, expressing each root in the form $r(\cos\theta+i\sin\theta)$ for some values of r and θ.
Show the roots on a single Argand diagram.

22. (a) Write down the binomial expansion of $(a+b)^4$.
(b) De Moivre's Theorem states that $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$, where n is an integer.
Starting with $\cos4\theta+i\sin4\theta=(\cos\theta+i\sin\theta)^4$, use a binomial expansion and equate real and imaginary parts to find identities for $\cos4\theta$ and $\sin4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Hence show that
$$\tan4\theta=\frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}.$$

23. (a) Write down the binomial expansion of $(a+b)^5$.
(b) Starting with $\cos5\theta+i\sin5\theta=(\cos\theta+i\sin\theta)^5$, use a binomial expansion and equate imaginary parts to show that
$$\sin5\theta=\sin\theta(5\cos^4\theta-10\cos^2\theta\sin^2\theta+\sin^4\theta).$$
Hence show that, if θ is not a multiple of π,
$$\frac{\sin5\theta}{\sin\theta}=16\cos^4\theta-12\cos^2\theta+1.$$

24. (a) Verify that $z=2-i$ is a root of the equation $z^4-3z^3+2z^2+z+5=0$ and write down another root of this equation.
(b) Hence solve the equation $z^4-3z^3+2z^2+z+5=0$ completely.

25. (a) Verify that $z=5+2i$ is a root of the equation $z^4-8z^3+6z^2+88z-87=0$ and write down another root of this equation.
(b) Hence solve the equation $z^4-8z^3+6z^2+88z-87=0$.

26. (a) Write the real number 8 in the form $r(\cos\theta+i\sin\theta)$, where $r>0$ and $-180°<\theta\le180°$.
(b) Hence, or otherwise, find the three roots of the equation $z^3=8$, expressing each root in the form $a+bi$ for real numbers a and b.
(c) Show that the sum of the three roots is zero, and find the product of the three roots.

27. (a) Show that $z=1+i$ is a root of the equation $z^4+z^3-2z^2+2z+4=0$ and write down another root of this equation.
(b) Hence solve the equation $z^4+z^3-2z^2+2z+4=0$ completely.

28. (a) Express the complex number $1+i$ in polar form and hence write down $(1+i)^6$ in the form $r(\cos\theta+i\sin\theta)$ for some values of r and θ.
(b) Express the complex number $\sqrt{3}-i$ in polar form and hence write down $(\sqrt{3}-i)^4$ in the form $r(\cos\theta+i\sin\theta)$ for some values of r and θ.
(c) Show that $\dfrac{(1+i)^6}{(\sqrt{3}-i)^4}=\dfrac{1}{4}(\sqrt{3}+i)$.

29. (a) Express -16 in the form $r(\cos\theta+i\sin\theta)$, where $r>0$ and $-180°<\theta\le180°$.
(b) Hence find the four roots of the equation $z^4=-16$, expressing each root in the form $r(\cos\theta+i\sin\theta)$ for some values of r and θ.
Show the four roots on a single Argand diagram.

30. Find z in the form $x+yi$, where x and y are real numbers, given that
$$\frac{1}{1+i}+\frac{1}{z}=\frac{1}{1-i}.$$

31. Find a real root of the equation $2z^3-3z+10=0$ and hence solve this equation completely.

32. One root of the equation $z^2+az+b=0$, where a and b are real constants, is $z=2+3i$.
Find the values of a and b.

33. (a) Verify that $z=2+i$ is a root of the equation $z^3-11z+20=0$ and write down another root of this equation.
(b) Hence solve the equation $z^3-11z+20=0$ completely.

34. (a) Show that $z = 1+i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$ and write down another root of this equation.
(b) Hence solve the equation $z^4 + 3z^2 - 6z + 10 = 0$ completely.

35. Solve the pair of simultaneous equations below for the complex numbers z and w, expressing each solution in the form $a+bi$ for real numbers a and b.

$$\begin{aligned} 4z + 3w &= 23 \\ z + iw &= 6+8i \end{aligned}$$

36. Repeat question 35 for the pair of simultaneous equations below.

$$\begin{aligned} (1+i)z + w &= 2-i \\ z + (1+i)w &= 6 \end{aligned}$$

37. (a) Express the complex number $1-i$ in polar form and hence write $(1-i)^{10}$ in the form $r(\cos\theta + i\sin\theta)$ for some values of r and θ.
(b) Express the complex number $\sqrt{3}i - 1$ in polar form and hence write $(\sqrt{3}i-1)^4$ in the form $r(\cos\theta + i\sin\theta)$ for some values of r and θ.
(c) Express $\dfrac{(1-i)^{10}}{(\sqrt{3}i-1)^4}$ in the form $x+yi$, where x and y are real numbers.

38. The equation $z^4 + 2z^3 - z^2 + az + 90 = 0$, where a is a real constant, has a root $z = 2-i$.
(a) Find the value of a.
(b) Find all the other solutions of this equation.

39. Let $z = \cos\theta + i\sin\theta$.

For any positive integer n, De Moivre's Theorem states that $z^n = \cos n\theta + i\sin n\theta$

Hence, if n is a positive integer,

$$\begin{aligned} \frac{1}{z^n} &= \frac{1}{\cos n\theta + i\sin n\theta} \\ &= \frac{1(\cos n\theta - i\sin n\theta)}{(\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)} \\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta - i^2\sin^2 n\theta} \\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} \\ &= \frac{\cos n\theta - i\sin n\theta}{1} \qquad [\text{since } \cos^2 n\theta + \sin^2 n\theta = 1] \\ &= \cos n\theta - i\sin n\theta, \end{aligned}$$

and so
$$\begin{aligned} z^n + \frac{1}{z^n} &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ &= 2\cos n\theta. \end{aligned}$$

This fact, along with the binomial theorem, allows us to find an identity for $\cos^n\theta$ in terms of cosines of multiples of θ.

First, note that when $n = 1$ we have $z + \frac{1}{z} = 2\cos\theta$.

Now $\quad (2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4$

$$\begin{aligned} \Rightarrow \quad 16\cos^4\theta &= z^4 + 4\cdot z^3\cdot\frac{1}{z} + 6\cdot z^2\cdot\left(\frac{1}{z}\right)^2 + 4\cdot z\cdot\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 \\ &= z^4 + 4\cdot z^3\cdot\frac{1}{z} + 6\cdot z^2\cdot\frac{1}{z^2} + 4\cdot z\cdot\frac{1}{z^3} + \frac{1}{z^4} \\ &= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} \\ &= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \\ &= 2\cos 4\theta + 4\cdot 2\cos 2\theta + 6 \\ &= 2\cos 4\theta + 8\cos 2\theta + 6 \\ \Rightarrow \quad \cos^4\theta &= \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6) \\ &= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} \end{aligned}$$

This expression allows us to integrate $\cos^4\theta$ with respect to θ as follows:

$$\begin{aligned} \int \cos^4\theta \; d\theta &= \int\left(\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}\right)d\theta \\ &= \frac{1}{8}\cdot\frac{1}{4}\sin 4\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta + \frac{3}{8}\theta + C \\ &= \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + C. \end{aligned}$$

(a) Starting with $(2\cos\theta)^2 = \left(z + \frac{1}{z}\right)^2$, show that $\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$.

Hence find $\int \cos^2\theta \, d\theta$.

(b) Starting with $(2\cos\theta)^3 = \left(z + \frac{1}{z}\right)^3$, show that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

Hence find $\int \cos^3\theta \; d\theta$.

(c) Starting with $(2\cos\theta)^5 = \left(z+\frac{1}{z}\right)^5$, show that

$$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta.$$

Hence find $\int \cos^5\theta \, d\theta$.

(d) Find $\int \cos^6\theta \, d\theta$.

ANSWERS

1. (a) $7-9i$ (b) $3+4i$ (c) i (d) $\frac{5}{17}+\frac{14}{17}i$
   (e) $-\frac{2}{5}-\frac{1}{5}i$ (f) $\frac{5}{13}+\frac{12}{13}i$ (g) $\frac{3}{10}-\frac{1}{10}i$ (h) $\frac{7}{5}+\frac{4}{5}i$

2. $z+\frac{25}{z}=6$

3. $z=\frac{1}{2}-\frac{1}{2}i$

4. (a) $z_1=\frac{3}{5}-\frac{4}{5}i$; $z_2=-\frac{3}{2}+\frac{1}{2}i$ (b) $5z_1+2z_2=-3i$; $5z_1-2z_2=6-5i$

5. $z=2-i$; $z^2=3-4i$

6. (a) $z=\frac{63}{25}+\frac{16}{25}i$ (b) $r=\frac{13}{5}$; $\theta=14.3°$

7. $2-11i$

8. (a) $z=-3-i$ (b) $r=\sqrt{10}$; $\theta=-161.6°$

9. $z=2\pm 4i$; modulus of each root $=2\sqrt{5}$, arguments are $\pm 63.4°$

10. $a=4$, $b=-5$

11. (a) $z_3=7-i$
    (b) (i) $r=\sqrt{2};\theta=-45°$ (ii) $r=5;\theta=36.9°$ (iii) $r=5\sqrt{2};\theta=-8.1°$
    (c) area of triangle ABC = 12 square units

12. (a) $z_1=3+i$; $z_2=2i$; $z_3=2-2i$

13. $\frac{5}{169}-\frac{12}{169}i$

15. (a) $w=-\frac{9}{25}-\frac{13}{25}i$ (b) modulus of $w=\frac{1}{5}\sqrt{10}$

16. $1+i=\sqrt{2}(\cos 45°+i\sin 45°)$, $1-i=\sqrt{2}\{\cos(-45°)+i\sin(-45°)\}$;
    $(1+i)^{20}+(1-i)^{20}=-2048$

17. $\sqrt{3}+i=2(\cos 30°+i\sin 30°)$; (a) $n=6$ (b) $n=3$

18. $z=-2\pm 2i$; $\frac{\alpha+\beta+4i}{\alpha\beta+8i}=\frac{1}{2}i$

19. $z = 3+3i$ or $z = 3-i$

20. $z = 2$; $z = 2$ or $z = 1 \pm 2i$

21. (a) $8i = 8(\cos 90° + i\sin 90°)$
(b) $z_1 = 2(\cos 30° + i\sin 30°)$; $z_2 = 2(\cos 150° + i\sin 150°)$; $z_3 = 2(\cos 270° + i\sin 270°)$

22. (a) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
(b) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$; $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

23. (a) $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

24. (a) $z = 2+i$ (b) $z = 2 \pm i$ or $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

25. (a) $z = 5-2i$ (b) $z = 5 \pm 2i$, $z = 1$ or $z = -3$

26. (a) $8 = 8(\cos 0° + i\sin 0°)$ (b) $z_1 = 2$; $z_2 = -1+\sqrt{3}i$; $z_3 = -1-\sqrt{3}i$
(c) product of roots = 8

27. (a) $z = 1-i$ (b) $z = 1 \pm i$, $z = -1$ or $z = -2$

28. (a) $1+i = \sqrt{2}(\cos 45° + i\sin 45°)$; $(1+i)^6 = 8(\cos 270° + i\sin 270°)$
(b) $\sqrt{3} - i = 2\{\cos(-30°) + i\sin(-30°)\}$; $(\sqrt{3}-i)^4 = 16\{\cos(-120°) + i\sin(-120°)\}$

29. (a) $-16 = 16(\cos 180° + i\sin 180°)$
(b) $z_1 = 2(\cos 45° + i\sin 45°)$; $z_2 = 2(\cos 135° + i\sin 135°)$;
$z_3 = 2(\cos 225° + i\sin 225°)$; $z_4 = 2(\cos 315° + i\sin 315°)$

30. $z = -i$

31. $z = -2$; $z = -2$ or $z = 1 \pm \frac{\sqrt{6}}{2}i$

32. $a = -4$, $b = 13$

33. (a) $z = 2-i$ (b) $z = 2 \pm i$ or $z = -4$

34. (a) $z = 1-i$ (b) $z = 1 \pm i$ or $z = -1 \pm 2i$

35. $z = 2+3i$, $w = 5-4i$

36. $z = 1+i$, $w = 2-3i$

37. (a) $1-i = \sqrt{2}\{\cos(-45°) + i\sin(-45°)\}$;
$(1-i)^{10} = 32\{\cos(-450°) + i\sin(-450°)\}$
(b) $\sqrt{3}i - 1 = 2(\cos 120° + i\sin 120°)$; $(\sqrt{3}-i)^4 = 16(\cos 480° + i\sin 480°)$
(c) $\dfrac{(1-i)^{10}}{(\sqrt{3}i-1)^4} = -\sqrt{3} + i$

38. (a) $a = -42$ (b) $z = 2+i$ or $z = -3 \pm 3i$

39. (a) $\int \cos^2\theta \, d\theta = \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C$
(b) $\int \cos^3\theta \, d\theta = \frac{1}{12}\sin 3\theta + \frac{3}{4}\sin\theta + C$
(c) $\int \cos^5\theta \, d\theta = \frac{1}{80}\sin 5\theta + \frac{5}{48}\sin 3\theta + \frac{5}{8}\sin\theta + C$
(d) $\int \cos^6\theta \, d\theta = \frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta + C$

COMPLEX NUMBERS HOMEWORK 1

1. Solve the equation $z^2 - 10z + 29 = 0$ for the complex number z.

2. Express in the form $x + yi$, where x and y are real numbers:

(a) $(3+2i)(1+4i)$ (b) $\dfrac{7+26i}{5+2i}$ (c) $\dfrac{1+8i}{5+i}$

(d) $\dfrac{9+7i}{(1+i)(2+i)}$ (e) $\dfrac{8-i}{1-2i}+\dfrac{7+6i}{2+i}$

3. Solve each equation below for the complex number z.

(a) $(3+i)z = -1+i$ (b) $\dfrac{z}{2-5i} = 2+i$

4. Find the modulus and argument of the complex number $-2\sqrt{3}+6i$.

5. Find the two roots of the equation $z^2 - 6z + 16 = 0$, expressing each solution in the form $x + yi$, where x and y are real numbers.

Find the modulus and argument (correct to the nearest $0.1°$) of each root.

6. Express the complex number $\dfrac{4}{1-i}$ in the form $x + yi$, where x and y are real numbers, and hence find the modulus and argument of $\dfrac{4}{1-i}$.

7. Let $z = -3 + \sqrt{3}i$.

Express z in polar form and hence, or otherwise, find z^4 in the form $x + yi$, where x and y are real numbers.

8. Let $z = 1 + \sqrt{3}i$ and $w = 1 + i$.

(a) Find the modulus and argument of z, and hence express z^5 in the form $r(\cos\theta + i\sin\theta)$ for some values of r and θ.

(b) Find the modulus and argument of w, and hence express w^6 in the form $r(\cos\theta + i\sin\theta)$ for some values of r and θ.

(c) Using your expressions in (a) and (b) above, find in the form $x + yi$, where x and y are real numbers:

(i) $z^5 w^6$ (ii) $\dfrac{z^5}{w^6}$

ADVANCED HIGHER MATHEMATICS

COMPLEX NUMBERS HOMEWORK 2

1. (a) Find the modulus and argument of the complex number $-2+2\sqrt{3}i$.

(b) Find the two complex numbers z for which $z^2=-2+2\sqrt{3}i$, expressing each solution in the form $r(\cos\theta+i\sin\theta)$.

Show the two solutions on the same Argand diagram.

2. (a) Express the complex number $8i$ in the form $r(\cos\theta+i\sin\theta)$, where $r>0$ and $-180°<\theta\leq 180°$.

(b) The cube roots of $8i$ are the three complex numbers z for which $z^3=8i$.

Express each of the cube roots of $8i$ in the form $x+yi$ for some real numbers x and y.

Hence show that the sum of the three cube roots of $8i$ is zero, and find the product of the three cube roots.

3. (a) Show that $z=3$ is a real root of the equation

$$z^3-z^2+4z-30=0,$$

and express $z^3-z^2+4z-30$ as the product of a linear factor and a quadratic factor with real coefficients.

(b) Hence find **all** the roots of the equation $z^3-z^2+4z-30=0$.

4. (a) **Given** that $z=2+i$ is a root of the equation

$$z^4-2z^3-z^2+2z+10=0,$$

write down another root of this equation.

(b) Hence find **all** the roots of the equation $z^4-2z^3-z^2+2z+10=0$.

5. (a) **Show** that $z=1+2i$ is a root of the equation

$$z^4-5z^3+13z^2-19z+10=0,$$

and write down another root of this equation.

(b) Hence find **all** the roots of the equation $z^4-5z^3+13z^2-19z+10=0$.

6. De Moivre's Theorem states that

$$\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n$$

for any integer n.

(a) Starting with

$$\cos 4\theta+i\sin 4\theta=(\cos\theta+i\sin\theta)^4,$$

show that $\cos 4\theta$ can be expressed in the form

$$\cos 4\theta=k\cos^4\theta+l\cos^2\theta\sin^2\theta+m\sin^4\theta$$

for some real constants k, l and m.

(b) Show also that $\sin 4\theta$ can be expressed in the form

$$\sin 4\theta=p\cos^3\theta\sin\theta+q\cos\theta\sin^3\theta$$

for some real constants p and q.

(c) Find an expression for $\cos 4\theta$ entirely in terms of $\cos\theta$.

7. Given the complex number $z=\cos\theta+i\sin\theta$, show that

$$z^n+\frac{1}{z^n}=2\cos n\theta$$

for any positive integer n.

(a) Starting with

$$(2\cos\theta)^5=\left(z+\frac{1}{z}\right)^5,$$

show that

$$\cos^5\theta=\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta.$$

(b) Hence find

$$\int\cos^5\theta . d\theta.$$

Please turn over for questions 8 and 9.

8. The complex number z moves in the complex plane subject to the restriction

$$|z + i| = 3 .$$

Find the equation of the locus of z and sketch the locus on an Argand diagram.

9. The complex number z moves in the complex plane subject to the restriction

$$|z - 3| = |z - 2i| .$$

Find the equation of the locus of z.